

# IR Finiteness of Fermion Loop Diagrams

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We show that a general fermion loop diagram is finite in both soft and collinear regions and therefore, it's IR finite.

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## I. INTRODUCTION

The structure of infrared singularities of Feynman graphs is very well understood at one loop level [1][2]. These infrared singularities appear due to the massless particles present in the theory. We may therefore call them “mass singularities” as well. In literature these mass singularities are classified as soft and collinear singularities. The appearance of soft singularity in loop diagrams is associated with the exchange of massless particles between two on-shell particles. On the other hand, splitting of a massless external particle into two massless internal particles of a loop diagram gives rise to collinear singularity. In fact, these are only the necessary conditions for a loop diagram to have soft or collinear divergence. Actual singularities appear as any of the momenta of internal lines, in these configurations, vanishes (soft divergence) or it becomes parallel to one of it's neighboring external legs (collinear divergence). Both of these singularities are logarithmic in  $D = 4$  dimensions.

A soft singular configuration of loop diagrams, with massless external as well as internal particles, contains two collinear configurations and this situation leads to the possibility of overlapping of soft and collinear singularities [7]. The overlapping singularity is also logarithmic in nature. In dimensional regularization ( $D = 4 - 2\epsilon_{IR}, \epsilon_{IR} \rightarrow 0^-$ ), IR divergence of a loop integral appears as  $1/\epsilon_{IR}$  poles. The IR divergent scalar one loop integrals have generic form given by  $\sim \frac{A}{\epsilon_{IR}^2} + \frac{B}{\epsilon_{IR}} + C + \mathcal{O}(\epsilon_{IR})$ , where  $A, B, C$  are complex functions of kinematic invariants [3]. The  $1/\epsilon_{IR}^2$  term corresponds to the overlapping of soft and collinear divergence. It should be obvious that unlike in the case of UV singularity, tensor integrals do not spoil the (logarithmic) structure

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of IR divergence of scalar integrals. In Ref.[4] an expression to determine  $A$  and  $B$  for a general  $N$ -point one loop tensor integral is obtained. A very naive way of understanding these mass singularities at one loop can be found in [5].

There are many scattering processes which, at one loop, proceed via fermion loop diagrams. In fact some of these scattering processes are possible only at one loop *i.e.* there is no tree level diagram for them [8]. A well known standard example is of four photon interaction in QED. Though these fermions in the loop are not exactly massless, at very high energies their masses can be neglected. As discussed above, working in vanishing fermion mass limit may lead to mass singularities. In this case, the singularities would show up as large-logs of vanishing fermion masses. This is another way of regularizing IR singularities, often called “mass regularization” in literature. In mass regularization, the generic form of fermion loop diagrams is,  $\sim A \ln^2(m^2) + B \ln(m^2) + C + \mathcal{O}(m^2)$  [9]. In this form,  $\ln^2(m^2)$  piece refers to overlapping singularity.

In the following we show that for a given fermion loop diagram,  $A = B = 0$ , that is, it is IR finite. The proof is obvious for the case of massive fermions in the loop. Also, if all the external legs are massive, the diagram would be IR finite even for massless fermions in the loop. So we need to consider only those fermion loop diagrams in which fermions are massless or more correctly their masses can be neglected and at least one external leg is massless.

## II. PROOF OF THE MAIN RESULT

In general we can have massless scalars, gauge bosons and gravitons attached to the fermion loop. With one massless external particle we expect only collinear singularity while for two adjacent external massless particles, soft and collinear and their overlap may develop. IR finiteness of a fermion loop diagram can be shown by showing it's soft finiteness and collinear finiteness. This automatically takes care of it's finiteness in overlapping regions. At one loop the general fermion loop integral has following form, fig.1.

$$I \simeq \int d^D l \frac{\text{tr}(\dots \not{l}_{i-1} V_i \not{l}_i V_{i+1} \not{l}_{i+1} \dots)}{\dots l_{i-1}^2 l_i^2 l_{i+1}^2 \dots}, \quad (1)$$

where  $V_i$ 's are vertex factors for a given massless external particle attached to the fermion loop.

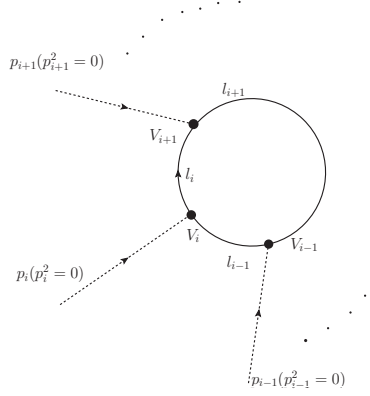


FIG. 1: Massless fermion loop diagram. Dotted external lines represent any suitable massless particle.

From momentum conservation at each vertex it is clear that  $l_i = l_{i-1} + p_i$ ,  $l_{i+1} = l_i + p_{i+1}$  etc. The Feynman rules for needed vertices are given in fig.2. Factors of ‘ $i$ ’ and coupling constants etc. are dropped in writing those rules, for simplification. Feynman rules for graviton-fermion interaction is derived in [6]. We did not consider two graviton-fermion vertex since it does not correspond to soft or collinear configuration described above. We closely follow the analysis of IR singular structure of one loop diagrams, given in Ref.[5].

First we would like to see the behavior of this integral as any of the internal lines becomes soft *i.e.* it’s momentum vanishes. Without loss of generality we consider softness of  $l_i$  and take  $l_i = \epsilon$ . We see that in the limit  $\epsilon \rightarrow 0$ , the denominators which vanish in general are,

$$\begin{aligned} l_{i-1}^2 &= \epsilon^2 - 2\epsilon \cdot p_i, \\ l_i^2 &= \epsilon^2 \quad \text{and} \\ l_{i+1}^2 &= \epsilon^2 + 2\epsilon \cdot p_{i+1}, \end{aligned} \tag{2}$$

where we have used on mass-shell conditions for  $p_i$  and  $p_{i+1}$ . Neglecting  $\epsilon^2$  with respect to  $\epsilon \cdot p_i$  and  $\epsilon \cdot p_{i+1}$  in Eq.(2), we see that the integral (1), in soft limit, behaves as

$$I \sim \int d^D \epsilon \frac{\not{\epsilon}}{\epsilon \cdot p_i \epsilon^2 \epsilon \cdot p_{i+1}} \sim \epsilon^{D-3} \tag{3}$$

and it vanishes in  $D = 4$  dimensions. Thus each fermion loop diagram is soft finite, independent of kind of massless external particles attached to it. We should mention here that soft finiteness of fermion loop diagrams is indirectly shown by Kinoshita in [1].

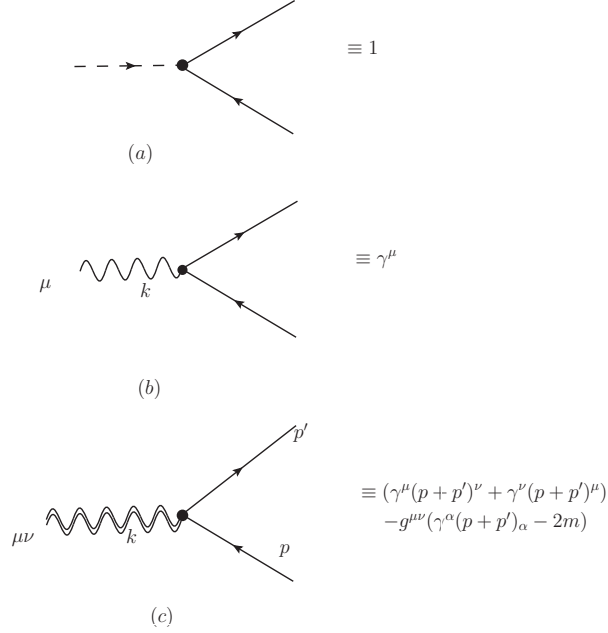


FIG. 2: Feynman rules: (a) scalar-fermion-fermion vertex (b) vector boson-fermion-fermion vertex (c) graviton-fermion-fermion vertex

Next we consider the behavior of fermion loop integral in collinear region. We take  $l_i = xp_{i+1} + \epsilon_\perp$ , ( $x \neq 0, -1$  as it corresponds to softness of  $l_i$  and  $l_{i+1}$  respectively) with  $\epsilon_\perp \cdot p_{i+1} = 0$ . Note that in  $\epsilon_\perp \rightarrow 0$  limit, this condition implies collinearity of  $l_i$  with  $p_{i+1}$ . In this collinear limit only vanishing denominators are  $l_i^2 = \epsilon_\perp^2$  and  $l_{i+1}^2 = \epsilon_\perp^2$ . Thus the integral in Eq. (1) reads,

$$I \simeq \int \frac{d^D \epsilon_\perp}{\epsilon_\perp^4} \text{tr}(\dots l_{i-1} V_i \not{p}_{i+1} V_{i+1} \not{p}_{i+1} \dots). \quad (4)$$

We need to make the above substitution for  $l_i$  in  $V_i$ 's also, in case they depend on the loop momentum *e.g.* in graviton-fermion vertex. Since the vertex factors are different for different kind of external particles, we will consider three separate cases to see the behavior of fermion loop integral in collinear limit. Referring to Feynman rules in fig. 2, we see

(i) *for scalars*,  $V_{i+1} = 1$ , therefore the integral in Eq. (4) is

$$I \simeq \int \frac{d^D \epsilon_\perp}{\epsilon_\perp^4} \text{tr}(\dots \not{l}_{i-1} \not{p}_{i+1} \not{p}_{i+1} \dots) = 0, \quad \text{since, } \not{p}_{i+1} \not{p}_{i+1} = p_{i+1}^2 = 0. \quad (5)$$

(ii) *for vector bosons,*

$$V_{i+1} = \gamma_\mu e_{i+1}^\mu = \not{\epsilon}_{i+1}. \quad (6)$$

$e_{i+1}^\mu$  is the polarization vector of the gauge boson with momentum  $p_{i+1}$ . Using transversality and on-shell condition for vector boson, we see that the

$$\not{p}_{i+1} \not{\epsilon}_{i+1} \not{p}_{i+1} = 2 \not{p}_{i+1} e_{i+1} \cdot p_{i+1} - p_{i+1}^2 \not{\epsilon}_{i+1} = 0. \quad (7)$$

and therefore the fermion loop integral in collinear limit, Eq. (4), vanishes. Finally

(iii) *for gravitons,*

$$V_{i+1} = (\gamma_\mu (2l_i + p_{i+1})_\nu + \gamma_\nu (2l_i + p_{i+1})_\mu - g_{\mu\nu} (2\not{l}_i + \not{p}_{i+1} - 2m)) e_{i+1}^{\mu\nu}, \quad (8)$$

where  $e_{i+1}^{\mu\nu}$  is polarization tensor for graviton of momentum  $p_{i+1}$ . It has following well known properties,

$$\begin{aligned} e_{i+1}^{\mu\nu} g_{\mu\nu} &= (e_{i+1})_\mu^\mu = 0 \text{ (traceless condition),} \\ p_{i+1}^\mu (e_{i+1})_{\mu\nu} &= 0 \text{ (transverse condition).} \end{aligned} \quad (9)$$

Using these properties the vertex factor in Eq. (8) becomes

$$V_{i+1} = 4\gamma_\mu e_{i+1}^{\mu\nu} (l_i)_\nu. \quad (10)$$

In the collinear limit, taken above, it is

$$V_{i+1} = 4x\gamma_\mu e_{i+1}^{\mu\nu} (p_{i+1})_\nu = 0, \quad (11)$$

due to transverse condition. Therefore the fermion loop diagram with external gravitons, like the cases of scalars and vector bosons, is also collinear finite. Combining all the above results of this section we see that a general fermion loop diagram is soft as well as collinear finite.

### III. CONCLUSION

In the above, we have shown that the most general possible fermion loop diagram is IR finite. We have seen that the soft finiteness of fermion loop diagrams follows from simple power counting in vanishing loop momentum while their collinear finiteness results, utilizing various properties of massless external particles attached to the loop. The result holds even for axial coupling of external particles with the fermion. Though we have not considered any flavor change in the loop, it should be clear from above analysis that our result remains true for any possible flavor changing interaction vertex in the loop. We would also like to emphasize that this result is important from the point of view of making numerical checks on codes for calculating fermion loop diagrams.

### IV. ACKNOWLEDGMENT

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  - [7] Note that at one loop, overlapping of two soft regions or two distant collinear regions does not take place in general.
  - [8] Such processes at one loop, are UV as well as IR finite and this is normally expected when contribution from all the relevant one loop diagrams are included.
  - [9] In general there is no one-to-one correspondence between results obtained using dimensional regularization and mass regularization.